Econometric Methods, 2023 **Assignment 10**

Due: 12/6, 0:00am

Problem 1

Let $(Y_t)_{1\le t\le T}$ be an IID sequence of random variables with the mean μ and the variance σ^2. Define the lag-k autocorrelation coefficient:

$$\rho(k) := \mathrm{corr}(Y_t, Y_{t-k})$$

and the lag-k sample autocorrelation coefficient:

$$\widehat{\rho}(k) := \frac{1}{T-k}\sum_{t=k+1}^{T}\Big(\frac{Y_t-\bar{Y}}{\widehat{\sigma}}\Big)\Big(\frac{Y_{t-k}-\bar{Y}}{\widehat{\sigma}}\Big),$$

where $\bar{Y} := \sum_{t=1}^{T} Y_t/T$ and $\widehat{\sigma}^2 := \sum_{t=1}^{T}(Y_t-\bar{Y})^2/T$, for some fixed k which is smaller than T. Please show that

$$T\widehat{\rho}^2(k) \xrightarrow{d} \chi^2(1),$$

as $T\to\infty$.

Solution: WLOG, assume that $\sigma^2>0$ and $\widehat{\sigma}^2>0$. First, note that $\rho(k)=0$ ($\because Y_t \sim$ IID),

$$\sqrt{T}(\widehat{\rho}(k)-\rho(k)) = \sqrt{T}\widehat{\rho}(k) = \frac{\sigma^2}{\widehat{\sigma}^2}\sqrt{\frac{T}{T-k}}\Big(\frac{1}{\sqrt{T-k}}\sum_{t=k+1}^{T}\widehat{u}_{t-k}\Big),$$

where $\widehat{u}_{t-k} := \widehat{Y}_t^*\widehat{Y}_{t-k}^* = Y_t^*Y_{t-k}^* + o_p(1)$, with $\widehat{Y}_t^* := (Y_t-\bar{Y})/\sigma$, and $Y_t^* = (Y_t-\mu)/\sigma$ such that $\widehat{Y}_t^* = Y_t^* + o_p(1)$. Let $u_{t-k} := Y_t^*Y_{t-k}^*$. Since that $\widehat{\sigma}^2 \xrightarrow{p} \sigma^2$, $T/(T-k)\to 1$, as $T\to\infty$, and

$$\frac{1}{\sqrt{T-k}}\sum_{t=k+1}^{T}\widehat{u}_{t-k} = \frac{1}{\sqrt{T-k}}\sum_{t=k+1}^{T}u_{t-k} + o_p(1),$$

it leaves to show that

$$\sum_{t=k+1}^{T} u_{t-k}/\sqrt{T-k} \xrightarrow{d} \mathcal{N}(0,1),\ \text{as } T\to\infty.$$

Note that $\mathbb{E}[u_t|\mathcal{F}_{t-1}] = \mathbb{E}[Y_{t+k}^*Y_t^*|\mathcal{F}_{t-1}] = \sigma^{-2}\,\mathbb{E}[(Y_{t+k}-\mu)(Y_t-\mu)|\mathcal{F}_{t-1}] = 0$ ($\because Y_t\sim$ IID), and $\mathbb{E}[u_tu_{t+k}|\mathcal{F}_{t-1}] = \mathbb{E}[u_t\,\mathbb{E}[u_{t+k}|\mathcal{F}_{t+k-1}]|\mathcal{F}_{t-1}] = 0$. That is, $(u_t,\mathcal{F}_{t-1})$ is a standard MDS with

$\mathbb{E}[u_t^2|\mathcal{F}_{t-1}] = \mathbb{E}[(Y_{t+k}^* Y_t^*)^2|\mathcal{F}_{t-1}] = \mathbb{E}[{Y^*}_{t+k}^2|\mathcal{F}_{t-1}]\,\mathbb{E}[{Y_t^*}^2|\mathcal{F}_{t-1}] = 1$. By MDS CLT, we have

$$\sum_{t=k+1}^{T} u_{t-k}/\sqrt{T-k} \xrightarrow{d} \mathcal{N}(0,1), \text{ as } T \to \infty.$$

Thus, $\sqrt{T}(\widehat{\rho}(k) - \rho(k)) = \sqrt{T}\widehat{\rho}(k) \xrightarrow{d} \mathcal{N}(0,1)$, as $T \to \infty$ by Slutsky's lemma and continuous mapping theorem (CMT), so that

$$T\widehat{\rho}^2(k) = \left(\sqrt{T}\widehat{\rho}(k)\right)^2 \xrightarrow{d} \chi^2(1), \text{ as } T \to \infty.$$

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Problem 2

Following #1, please show that the "Box-Pierce statistic:"

$$Q(m) := T\sum_{k=1}^{m} \widehat{\rho}^2(k)$$

has the asymptotic distribution:

$$Q(m) \xrightarrow{d} \chi^2(m),$$

as $T \to \infty$.

Solution: Following Problem #1. Let $Q(m) := T(\widehat{\boldsymbol{\rho}} - \boldsymbol{\rho})'(\widehat{\boldsymbol{\rho}} - \boldsymbol{\rho})$, such that $\widehat{\boldsymbol{\rho}} := (\widehat{\rho}(1), \cdots, \widehat{\rho}(m))'$ and $\boldsymbol{\rho} := (\rho(1), \cdots, \rho(m))' = \mathbf{0}$. Note that

$$\sqrt{T}(\widehat{\boldsymbol{\rho}} - \boldsymbol{\rho}) = \sqrt{T}\widehat{\boldsymbol{\rho}} \xrightarrow{d} \mathcal{N}(0, \Omega),$$

where

$$\begin{aligned}\Omega &:= \lim_{T\to\infty} \text{Var}[\sqrt{T}(\widehat{\boldsymbol{\rho}} - \boldsymbol{\rho})] \\ &= \begin{pmatrix} \gamma^*(0) & \gamma^*(1) & \cdots & \gamma^*(m-1) \\ \gamma^*(1) & \gamma^*(0) & & \gamma^*(m-2) \\ \vdots & & \ddots & \vdots \\ \gamma^*(m-1) & \gamma^*(m-2) & \cdots & \gamma^*(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix} =: \boldsymbol{I}_m,\end{aligned}$$

since $\gamma^*(0) := \mathbb{E}[u_t^2] = \rho^2(0) = 1\ \forall k$, and $\gamma^*(j) := \mathbb{E}[u_{t-k}u_{t-l}] = \mathbb{E}[u_{t-k}]\,\mathbb{E}[u_{t-l}] = 0$ for $j := |k-l| = 1, \cdots, m-1$ ($\because Y_t \sim$ IID). Therefore, $Q(m)$ is the sum of the squares of m independent standard normal distributions, asymptotically,

$$Q(m) = (\sqrt{T}(\widehat{\boldsymbol{\rho}} - \boldsymbol{\rho}))'(\sqrt{T}(\widehat{\boldsymbol{\rho}} - \boldsymbol{\rho})) \xrightarrow{d} \chi^2(m), \text{ as } T \to \infty.$$

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Problem 3

Consider the following variables defined in Equity_Premium.csv:

① y : Equity premium (current)

② x_dfy : Default yield spread (lagged)

③ x_infl : Inflation (lagged)

④ x_svar : Stock variance (lagged)

⑤ x_tms : Term spread (lagged)

⑥ x_tbl : Treasury-bill rate (lagged)

⑦ x_dfr : Default return spread (lagged)

⑧ x_dp : Dividend price ratio (lagged)

⑨ x_ltr : Long-term return (lagged)

⑩ x_ep : Earnings price ratio (lagged)

⑪ x_bmr : Book to market (lagged)

⑫ x_ntis : Net equity expansion (lagged)

Please plot the sample autocorrelation function: $\{\widehat{\rho}(k)\}_{k=1}^{24}$ and the 95% asymptotic confident interval of $\rho(k)$, for $k = 1, 2, \cdots, 24$, in the same figure for each of the 12 time series. In addition, please test the null of IIDness for each of these time series using the Box-Pierce test statistic $Q(m)$, for $m = 12$ or 24, at the 5% level.

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