Advanced Mathematical Statistics (I), 2022 **Assignment 4**

Due: 12/31 23:59

Problem 1.

(a) Let $(\xi_i)_{1\leqslant i\leqslant n}$ be a random sample with finite means and variances, and $\mathbb{E}(\xi_1^{-1})$ exists, does $(n^{-1}\sum_{i=1}^n \xi_i^{-1})^{-1}$ converge almost surely? If so, to what?

(b) Consider ξ_n is a random variable from the binomial distribution $\mathrm{Bin}(n,\theta)$, where $\theta\in(0,1)$. Let

$$Y_n := \begin{cases} \log(\xi_n/n), & \xi_n \geqslant 1 \\ 1, & \xi_n = 0. \end{cases}$$

Show that Y_n converges to $\log\theta$ almost surely, and find the limiting distribution of

$$\sqrt{n}(Y_n - \log\theta) \xrightarrow{d} ?$$

(c) Let $(\xi_i)_{1\leqslant i\leqslant n}$ be a random sample from the uniform distribution on the interval $(\theta-1/2,\theta+1/2)$, where $\theta\in\mathbb{R}$ is unknown. Let $X_{(k)}$ be the kth order statistic of $(\xi_i)_{1\leqslant i\leqslant n}$. Show that $(X_{(1)}+X_{(n)})/2$ converges to θ almost surely.

Solution:

(a) Yes. By strong law of large numbers (SLLN), $n^{-1}\sum_{i=1}^n \xi_i^{-1} \xrightarrow{a.s.} \mathbb{E}(\xi_i^{-1})$, and by continuous mapping theorem (CMT), we have $(n^{-1}\sum_{i=1}^n \xi_i^{-1})^{-1} \xrightarrow{a.s.} \mathbb{E}(\xi_i^{-1})^{-1}$.

(b) Since that ξ_n can be decomposed by $\sum_{j=1}^n X_j$, where (X_j) are i.i.d. distributed with $\mathbb{P}(X_1=1)=\theta$ and $\mathbb{P}(X_1=0)=1-\theta$. By SLLN,

$$\xi_n/n = (\xi_n/n)\mathbf{I}(\xi_n\neq 0) + (\xi_n/n)\mathbf{I}(\xi_n=0) \xrightarrow{a.s.} \theta+0=\theta, \text{ as } n\to\infty;$$

Then, the desired result follows from the CMT immediately by the continuity of the log function on $(0,\infty)$,

$$Y_n \xrightarrow{a.s.} \log\theta, \text{as } n\to\infty.$$

By CLT, and the delta-method with $g(t)=\log t$ and $g'(t)=t^{-1}$,

$$\begin{aligned}\sqrt{n}(Y_n-\log\theta) &= \sqrt{n}(\log(\xi_n/n)\mathbf{I}(\xi_n\neq 0)-\log\theta) \\ &\xrightarrow{d} \mathcal{N}(0, g'(\theta)^2\theta(1-\theta)) \overset{d}{=} \mathcal{N}(0,(1-\theta)/\theta), \text{ as } n\to\infty.\end{aligned}$$

(c) For given any $\epsilon>0$,

$$\mathbb{P}(|X_{(1)}-(\theta-1/2)|>\epsilon) = \mathbb{P}(X_{(1)}>\epsilon+(\theta-1/2)) = \mathbb{P}(X_1>\epsilon+\theta-1/2)^n = (1-\epsilon)^n,$$

and similarly,

$$\mathbb{P}(|X_{(n)}-(\theta+1/2)|>\epsilon) = (1-\epsilon)^n.$$

Since that $\sum_{n=1}^{\infty}(1-\epsilon)^n < \infty$, we have that $X_{(1)} \xrightarrow{a.s.} (\theta - 1/2)$ and $X_{(n)} \xrightarrow{a.s.} (\theta + 1/2)$ by Borel-Cantelli lemma. Thus, $(X_{(1)} + X_{(n)})/2 \xrightarrow{a.s.} \theta$ by CMT.

□

Problem 2.

Let $(Y_n)_{n\geqslant 1}$ be a sequence of independent and identically distributed random variables with mean $\mathbb{E}\, Y_1 = \mu$ and finite variances $\operatorname{Var}(Y_1) = \sigma^2$. Define $(X_n)_{n\geqslant 2}$ as

$$X_n := \frac{Y_1Y_2 + Y_2Y_3 + \ldots + Y_{n-1}Y_n + Y_nY_1}{n}, \quad n = 2, 3, \ldots$$

Show that X_n converges to μ^2 in probability.

Solution: Let $Z_i := Y_iY_{i+1}$, $i = 1, \ldots, n-1$, $Z_n := Y_nY_1$. Then,

$$X_n = \frac{1}{n}\sum_{i=1}^{n} Z_i.$$

Since that $\mathbb{E}\, Z_1 = \mathbb{E}\, Y_1 \,\mathbb{E}\, Y_2 = \mu^2$, $\operatorname{Var}(Z_1) = \mathbb{E}(Y_1^2Y_2^2) - (\mathbb{E}\, Y_1Y_2)^2 = \sigma^4 + 2\sigma^2\mu^2$, and $\operatorname{cov}(Z_1, Z_2) = \mathbb{E}[Y_1Y_2^2Y_3] - \mathbb{E}[Y_1Y_2]\,\mathbb{E}[Y_2Y_3] = \mu^2(\mu^2+\sigma^2) - \mu^4 = \mu^2\sigma^2$.

$$\begin{aligned}\mathbb{E}\, X_n &= \frac{\mathbb{E}\, Z_1 + \mathbb{E}\, Z_2 + \ldots + \mathbb{E}\, Z_{n-1} + \mathbb{E}\, Z_n}{n} \\ &= \frac{n\mu^2}{n} = \mu^2,\end{aligned}$$

and

$$\begin{aligned}\operatorname{Var}(X_n) &= \frac{1}{n^2}\Big[n\operatorname{Var}(Z_1) + 2\sum_{i<j}\operatorname{cov}(Z_iZ_j)\Big] \\ &= \frac{n(\sigma^4 + 2\sigma^2\mu^2) + 2n\mu^2\sigma^2}{n^2} = \frac{4\mu^2\sigma^2 + \sigma^4}{n} \\ &\to 0, \text{ as } n \to \infty.\end{aligned}$$

Then, $X_n \to \mu^2$ in mean-squares sense, so that $X_n \xrightarrow{p} \mu^2$, as $n \to \infty$. (Or, using Chebyshev's inequality directly.)

□

Problem 3.

Consider an intercept-only model,

$$y_i = \mu + e_i, \; e_i \sim \text{i.i.d. } \mathcal{N}(0, \sigma^2), \; \sigma^2 > 0.$$

Define the sample skewness statistic as

$$\widehat{Sk} := \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\hat{e}_i}{\hat{\sigma}}\right)^3,$$

where $\hat{e}_i := y_i - \bar{y}$, $i = 1, \ldots, n$ and $\hat{\sigma}^2 := \sum_{i=1}^{n}\hat{e}_i^2/n$. Please show that

$$\sqrt{n}\widehat{Sk} \xrightarrow{d} \mathcal{N}(0, 6), \text{ as } n \to \infty.$$

Solution: WLOG, assume $\sigma > 0$.

$$\begin{aligned}
\hat{\sigma}^2 &= \sum_{i=1}^n \hat{e}_i^2/n \\
&= \sum_{i=1}^n e_i^2/n + 2\sum_{i=1}^n e_i(\hat{e}_i - e_i)/n + \sum_{i=1}^n (\hat{e}_i - e_i)^2/n \\
&= \sum_{i=1}^n e_i^2/n - 2\sum_{i=1}^n e_i(\bar{y} - \mu)/n + \sum_{i=1}^n (\bar{y} - \mu)^2/n \\
&\xrightarrow{p} \sigma^2
\end{aligned}$$

by the facts that $\sum_{i=1}^n e_i/n \xrightarrow{p} \mathbb{E}\, e_i = 0$, $\sum_{i=1}^n e_i^2/n \xrightarrow{p} \mathbb{E}\, e_i^2 = \sigma^2$, and $(\bar{y} - \mu) \xrightarrow{p} 0$ from WLLN. Then, $\hat{\sigma}^{-3} = (\hat{\sigma}^2)^{-3/2} \xrightarrow{p} (\sigma^2)^{-3/2} = \sigma^{-3}$ by continuous mapping theorem (CMT), and $\sqrt{n}\widehat{Sk} = \hat{\sigma}^{-3}(\sum_{i=1}^n \hat{e}_i^3/\sqrt{n}) \xrightarrow{p} \sigma^{-3}(\sum_{i=1}^n \hat{e}_i^3/\sqrt{n})$. It leaves to show that $\sum_{i=1}^n \hat{e}_i^3/\sqrt{n} \xrightarrow{d} \mathcal{N}(0, 6\sigma^6)$, so that the desired result follows by Slutsky's theorem.

Claim. $\sum_{i=1}^n \hat{e}_i^3/\sqrt{n} \xrightarrow{d} \mathcal{N}(0, 6\sigma^6)$.

Since that $\hat{e}_i = e_i + (\hat{e}_i - e_i)$ and $\hat{e}_i - e_i = -(\bar{y} - \mu)$, then

$$\begin{aligned}
\sum_{i=1}^n \hat{e}_i^3/\sqrt{n} &= \sum_{i=1}^n {e_i}^3/\sqrt{n} + 3\sum_{i=1}^n {e_i}^2(\hat{e}_i - e_i)/\sqrt{n} + 3\sum_{i=1}^n e_i(\hat{e}_i - e_i)^2/\sqrt{n} + \sum_{i=1}^n (\hat{e}_i - e_i)^3/\sqrt{n} \\
&=: \sum_{i=1}^n {e_i}^3/\sqrt{n} + (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}),
\end{aligned}$$

where (I) $\xrightarrow{p} -3\sum_{i=1}^n \sigma^2 e_i/\sqrt{n}$ since

$$\begin{aligned}
\sum_{i=1}^n {e_i}^2(\hat{e}_i - e_i)/\sqrt{n} &= -(\sum_{i=1}^n e_i^2/n)\left[\sqrt{n}(\bar{y} - \mu)\right] \\
&= -(\sum_{i=1}^n e_i^2/n)\left[\sqrt{n}(\sum_{i=1}^n (y_i - \mu)/n)\right] \\
&= -(\sum_{i=1}^n e_i^2/n)\sum_{i=1}^n e_i/\sqrt{n} \\
&\xrightarrow{p} -\sigma^2 \sum_{i=1}^n e_i/\sqrt{n}
\end{aligned}$$

by $\sum_{i=1}^n e_i^2/n \xrightarrow{p} \mathbb{E}\, e_i^2 = \sigma^2$ from WLLN;
(II) $\xrightarrow{p} 0$ since

$$\sum_{i=1}^n e_i(\hat{e}_i - e_i)^2/\sqrt{n} = (\sum_{i=1}^n e_i/\sqrt{n})(\bar{y} - \mu)^2 \xrightarrow{p} 0$$

by $(\bar{y} - \mu) \xrightarrow{p} 0$ from WLLN, and $\sum_{i=1}^n e_i/\sqrt{n} \xrightarrow{d} \mathcal{N}(0, \sigma^2)$ from CLT; and similarly,
(III) $\xrightarrow{p} 0$ since

$$\sum_{i=1}^n (\hat{e}_i - e_i)^3/\sqrt{n} = \left[\sqrt{n}(\bar{y} - \mu)\right]^2 (\bar{y} - \mu)/\sqrt{n} \xrightarrow{p} 0$$

by $1/\sqrt{n} \to 0$, $(\bar{y}-\mu) \xrightarrow{p} 0$ from WLLN, and $\sqrt{n}(\bar{y}-\mu) \xrightarrow{d} \mathcal{N}(0,\sigma^2)$ from CLT. Then,

$$\sum_{i=1}^{n} \hat{e}_i^3/\sqrt{n} \xrightarrow{p} \sum_{i=1}^{n} (e_i^3 - 3\sigma^2 e_i)/\sqrt{n}.$$

Also,

$$\sum_{i=1}^{n} (e_i^3 - 3\sigma^2 e_i)/\sqrt{n} \xrightarrow{d} \mathcal{N}(0, 6\sigma^6),$$

since $\mathbb{E}[e_i^3 - 3\sigma^2 e_i] = 0$, and by CLT,

$$\sum_{i=1}^{n} (e_i^3 - 3\sigma^2 e_i)/\sqrt{n} \xrightarrow{d} \mathcal{N}(0, V),$$

where $V = \text{Var}[e_i^3 - 3\sigma^2 e_i] = \mathbb{E}\, e_i^6 - 6\sigma^2\, \mathbb{E}\, e_i^4 + 9\sigma^4\, \mathbb{E}\, e_i^2 = 15\sigma^6 - 18\sigma^6 + 9\sigma^6 = 6\sigma^6$. Hence,

$$\sum_{i=1}^{n} \hat{e}_i^3/\sqrt{n} \xrightarrow{d} \mathcal{N}(0, 6\sigma^6).$$

We end the proof of **Claim**. Next, by Slutsky's theorem,

$$\begin{aligned}\sqrt{n}\widehat{Sk} &= \hat{\sigma}^{-3}\Big(\sum_{i=1}^{n} \hat{e}_i^3/\sqrt{n}\Big) \\ &\xrightarrow{d} \sigma^{-3}\mathcal{N}(0, 6\sigma^6) \stackrel{d}{=} \mathcal{N}(0, 6), \text{ as } n \to \infty.\end{aligned}$$

We are done.

□

Problem 4.

Let $\{(X_i, Y_i)\}_{1 \leqslant i \leqslant n}$ be a random sample from a bivariate normal density with correlation coefficient $\rho \in [0, 1)$. We denote $r_n := \sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})/\sqrt{\sum_{i=1}^{n}(X_i - \bar{X})^2 \sum_{i=1}^{n}(Y_i - \bar{Y})^2}$ as the corresponding sample correlation coefficient, providing that $\sqrt{n}(r_n - \rho) \xrightarrow{d} \mathcal{N}(0, (1-\rho^2)^2)$.
(a) For $\rho \neq 0$, please find the limiting distribution of

$$\sqrt{n}\Big(\frac{1}{2}\log(\frac{1+r_n}{1-r_n}) - \frac{1}{2}\log(\frac{1+\rho}{1-\rho})\Big) \xrightarrow{d} ?$$

as $n \to \infty$.
(b) Let $g(\rho) := (\exp(\rho) + \exp(-\rho))/2$. For $\rho = 0$, please find the limiting distribution of

$$2n[g(r_n) - g(\rho)] \xrightarrow{d} ?$$

as $n \to \infty$.

Solution:

(a) $\sqrt{n}\Big(\frac{1}{2}\log(\frac{1+r_n}{1-r_n}) - \frac{1}{2}\log(\frac{1+\rho}{1-\rho})\Big) \xrightarrow{d} \mathcal{N}(0, 1)$ as $n \to \infty$, by the first-order delta method.

(b) By the second-order delta method, you can obtain that $2n[g(r_n) - g(0)] \xrightarrow{d} \chi^2(1)$, as $n \to \infty$.

□

Problem 5.

Let $(\xi_n)_{n\geqslant 1}$ be independent and identically distributed with uniform distribution $\mathcal{U}[0,1]$, and $(Y_i)_{i\geqslant 1}$ be independent and identically exponential distributed, $\mathbb{P}(Y_1 > y) = \exp(-\lambda y)\mathbf{I}_{(0,\infty)}(y)$. Let $\xi_{(1)} \leqslant \xi_{(2)} \leqslant \ldots \leqslant \xi_{(n)}$ be the order statistics of the random sample $(\xi_1, \ldots, \xi_n)$, and $Y_{(1)} \leqslant Y_{(2)} \leqslant \ldots \leqslant Y_{(n)}$ be the order statistics of the random sample $(Y_1, \ldots, Y_n)$. We set $\xi_{(0)} = Y_{(0)} = 0$.

(a) Define $X_n := (\Pi_{i=1}^n \xi_i)^{-1/n}$. Show that

$$\sqrt{n}(X_n - e) \xrightarrow{d} \mathcal{N}(0, e^2), \text{ as } n \to \infty.$$

(b) Given $T_k := Y_1 + \ldots + Y_k$, show that the random vectors $(T_1/T_{n+1}, \ldots, T_n/T_{n+1})$ and $(\xi_{(1)}, \ldots, \xi_{(n)})$ have the same distribution.

(c) Let $D_i := (n-i+1)(Y_{(i)} - Y_{(i-1)})$, $i = 1, \ldots, n$. Show that $D_1, D_2, \ldots, D_n$ are independent, and D_i and Y_i have the same distribution, $i = 1, \ldots, n$.

(d) Let $R_n := \xi_{(n)} - \xi_{(1)}$ be the range of the random sample $(\xi_1, \ldots, \xi_n)$. What is the distribution of R_n? What is the **limiting** distribution of $2n(1 - R_n)$?

(e) Let $V_n := (\xi_{(1)} + \xi_{(n)})/2$, what is the conditional distribution of $V_n | R_n = r$?

(f) If $i < j$, find an expression for the conditional density of $\xi_{(j)}$ given $\xi_{(i)}$.

Solution:

(a) Let $Z_i := -\log \xi_i$, which follows the exponential distribution with $\mathbb{E}\, Z_1 = 1$ and $\text{Var}(Z_1) = 1$. Define $\bar{Z}_n := \sum_{i=1}^n Z_i/n$. By CLT,

$$\sqrt{n}(\bar{Z}_n - 1) \xrightarrow{d} \mathcal{N}(0, 1), \text{ as } n \to \infty.$$

Let $g(\theta) := \exp(\theta)$, with $g'(\theta) = \exp(\theta)$. Then, $X_n := (\Pi_{i=1}^n \xi_i)^{-1/n} = g(\bar{Z}_n)$, by delta method, we have

$$\begin{aligned}\sqrt{n}(X_n - e) = \sqrt{n}(g(\bar{Z}_n) - g(1)) &\xrightarrow{d} \mathcal{N}(0, g'(1)^2) \\ &\stackrel{d}{=} \mathcal{N}(0, e^2), \text{ as } n \to \infty.\end{aligned}$$

(b) First note that the joint density of $(\xi_{(1)}, \ldots, \xi_{(n)})$ is

$$f_{\xi_{(1)},\ldots,\xi_{(n)}}(x_1, \ldots, x_n) = n!\mathbf{I}(0 < x_1 < \ldots < x_n < 1),$$

and the joint density of $(Y_1, \ldots, Y_n)$ is

$$f_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n) = \lambda^n e^{-\lambda \sum_{i=1}^n y_i}\mathbf{I}(0 < y_1, \ldots, y_n < \infty).$$

Since that $Y_k = T_k - T_{k-1}$, $k = 1, \ldots, n+1$, $T_0 = 0$,

$$\begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ \vdots \\ Y_{n+1} \end{pmatrix} \begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ 0 & -1 & 1 & & \\ \vdots & 0 & -1 & 1 & \\ 0 & 0 & \cdots & -1 & 1 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ \vdots \\ \vdots \\ T_{n+1} \end{pmatrix},$$

we have the joint density of $(T_1, \ldots, T_{n+1})$ as

$$g_{T_1,\ldots,T_{n+1}}(t_1, \ldots, t_{n+1}) = f_{Y_1,\ldots,Y_{n+1}}(y_1, \ldots, y_{n+1})|J| = \lambda^{n+1} e^{-\lambda t_{n+1}},$$

in which the Jacobian determinant $J := |\partial(y_1, \ldots, y_{n+1})/\partial(t_1, \ldots, t_{n+1})| = 1$.
Similarly, let $U_k := T_k/T_{n+1}$, $k = 1, \ldots, n$, we have the joint density of $(U_1, \ldots, U_n, T_{n+1})$ as

$$h_{U_1,\ldots,U_n,T_{n+1}}(u_1, \ldots, u_n, t_{n+1}) = g_{T_1,\ldots,T_{n+1}}(t_1, \ldots, t_{n+1})|J| = \lambda^{n+1} e^{-\lambda t_{n+1}} t_{n+1}^n,$$

where

$$\begin{aligned} J &:= \left|\frac{\partial(t_1, \ldots, t_{n+1})}{\partial(u_1, \ldots, u_n, t_{n+1})}\right| \\ &= \det \begin{pmatrix} t_{n+1} & 0 & & & u_1 \\ 0 & t_{n+1} & & & u_2 \\ & & \ddots & 0 & \vdots \\ \vdots & & & t_{n+1} & u_n \\ 0 & \cdots & & 0 & 1 \end{pmatrix} = t_{n+1}^n, \end{aligned}$$

then

$$\begin{aligned} h_{U_1,\ldots,U_n}(u_1, \ldots, u_n) &= n! \int_0^\infty \frac{\lambda^{n+1} t_{n+1}^n}{n!} e^{-\lambda t_{n+1}} dt_{n+1} \\ &= n! \int_0^\infty f_{T_{n+1}}(t_{n+1}) dt_{n+1} \\ &= n!, \ 0 < u_1 < \ldots < u_n < 1. \end{aligned}$$

Thus, $(U_1, \ldots, U_n) \overset{d}{=} (\xi_{(1)}, \ldots, \xi_{(n)})$.
(c) The joint density of $(Y_{(1)}, \ldots, Y_{(n)})$ is

$$f_{Y_{(1)},\ldots,Y_{(n)}}(s_1, \ldots, s_n) = n! \lambda^n \exp(-\lambda \sum_{i=1}^n s_i) \mathbf{I}(0 < s_1 < s_2 < \ldots < s_n < \infty).$$

Let $\mathbf{D} := (D_1, \ldots, D_n)^\top$ and $\mathbf{Y} := (Y_{(1)}, \ldots, Y_{(n)})^\top$,

$$\mathbf{D} = A\mathbf{Y},$$

whose

$$A = \begin{pmatrix} n & & & & \\ -(n-1) & (n-1) & & & \\ 0 & -(n-2) & (n-2) & & \\ \vdots & 0 & 0 & & \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

Since that $D_1 + \ldots + D_n = Y_{(1)} + \ldots + Y_{(n)}$, we have the joint density of $(D_1, \ldots, D_n)$ is, with the Jacobian determinant $J := |\partial(s_1,\ldots,s_n)/\partial(d_1,\ldots,d_n)| = |A|^{-1} = 1/n!$,

$$\begin{aligned} g_{D_1,\ldots,D_n}(d_1,\ldots,d_n) &= f(s_1,\ldots,s_n)|J| \\ &= \lambda^n \exp(-\lambda \sum_{i=1}^n d_i) \\ &= (\lambda\exp(-\lambda d_1))(\lambda\exp(-\lambda d_2))\ldots(\lambda\exp(-\lambda d_n)) \\ &=: g(d_1)\ldots g(d_n). \end{aligned}$$

Hence, D_i are i.i.d. exponentially distributed as well as Y_i, $i = 1,\ldots,n$.

(d) The density of R_n is

$$f_{R_n}(r) = n(n-1)r^{(n-2)}(1-r)\mathbf{I}(0<r<1),$$

and then you can obtain the density of $2n(1-R_n)$ is,

$$\begin{aligned} h(r) &= \frac{n-1}{4n} r(1-r/2n)^{(n-2)}\mathbf{I}(0<r<2n) \\ &\to \frac{1}{4} re^{-r/2}\mathbf{I}_{(0,\infty)}(r), \text{ as } n\to\infty, \end{aligned}$$

since $\lim_{n\to\infty}(1-r/2n)^{(n-2)} = \exp(-r/2)$. So, $2n(1-R_n) \xrightarrow{d} \chi^2(4)$, as $n\to\infty$.
Alternatively, you can go another way. since that

$$\begin{aligned} \mathbb{P}(n\xi_{(1)} > x, n(1-\xi_{(n)}) > y) &= (1-y/n-x/n)^n \\ &\to \exp(-(x+y)) \\ &= \exp(-x)\exp(-y), \ x,y>0, \text{ as } n\to\infty, \end{aligned}$$

which implies that

$$\begin{pmatrix} n\xi_{(1)} \\ n(1-\xi_{(n)}) \end{pmatrix} \xrightarrow{d} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix},$$

where Z_1, Z_2 both follow the standard exponential distribution with unit mean. Then,

$$2n(1-R_n) = 2\Big[n(1-\xi_{(n)}) + n\xi_{(1)}\Big] \xrightarrow{d} \chi^2(4), \text{ as } n\to\infty.$$

(e) Following Example 5.4.7., you can calculate the joint density of (V_n, R_n) is

$$f_{V_n,R_n}(v,r) = n(n-1)r^{(n-2)}, \ 0<r<1, \ r/2<v<1-r/2.$$

Then the conditional density of $V_n|R_n = r$ is obtained as

$$f_{V_n|R_n=r}(v|r) = \frac{n(n-1)r^{(n-2)}}{n(n-1)r^{(n-2)}(1-r)} = \frac{1}{1-r}\mathbf{I}(r/2<v<1-r/2).$$

(f)

$$f_{\xi_{(j)}|\xi_{(i)}}(s|t) = \frac{(n-i)!}{(j-i-1)!(n-j)!}\left[\frac{s-t}{1-t}\right]^{(j-i-1)}\frac{1}{1-t}\left[\frac{1-s}{1-t}\right]^{n-j}\mathbf{I}(0<t<s\leqslant 1).$$

$\square$

Problem 6.

Textbook Exercises: 5.3, 5.6, 5.12, 5.23, 5.24, 5.36, 5.41, 5.44.
